

A probability loophole in the CHSH.

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In the present paper a robustness stress-test of the CHSH experiments for Einstein locality and causality is designed and employed. Random A and B from dice and coins, but based on a local model, run "parallel" to a real experiment. We found a local causal model with a nonzero probability to violate the CHSH inequality for some relevant quartets \mathcal{Q} of settings in the series of trials.

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I. INTRODUCTION & MODEL

The statistical basis of CHSH conclusion is studied. In 1964 J.S. Bell postulated the local hidden variables (LHV) correlation $E(a, b) = \int_{\lambda \in \Lambda} \rho_\lambda A_\lambda(a) B_\lambda(b) d\lambda$. Here, $A_\lambda(a)$ and $B_\lambda(b) \in \{-1, 1\}$. For more details see [1]. Clauser [3] derived the CHSH inequality $|S| \leq 2$ thereof with

$$S = E(1_A, 1_B) - E(1_A, 2_B) - E(2_A, 1_B) - E(2_A, 2_B). \quad (1)$$

In the CHSH, setting pairs $\mathcal{Q} = \mathcal{A} \times \mathcal{B}$ are used with $a \in \{1_A, 2_A\} = \mathcal{A}$ and $b \in \{1_B, 2_B\} = \mathcal{B}$. The $|S| \leq 2$ must be valid for all LHV models for each trial, at any moment. Our settings are the violating pairs $1_A = (1, 0, 0)$, $2_A = (0, 1, 0)$, $1_B = \frac{1}{\sqrt{2}}(1, -1, 0)$ and $2_B = \frac{1}{\sqrt{2}}(-1, -1, 0)$.

If $|S| > 2$ is found in nature such as in [4], then under local conditions, Einstein locality [2] does not occur. Of course, this is valid only if $|S| > 2$ is LHV impossible. We reformulate this as $\Pr\{|S| > 2 \mid \text{using LHV}\} = 0$. This must be true for each relevant \mathcal{Q} in each experiment for each model in order for a CHSH experiment to make sense.

In the proposed stress-test, Alice and Bob *simulate* a "parallel" independent A and B sequence with additional coins and dices [5]. Let us define 3 sets: $\Omega_+(a, b, x, y) := \{\lambda \in \Lambda \mid A_\lambda(a) B_\lambda(b) = A_\lambda(x) B_\lambda(y) = +1\}$, $\Omega_-(a, b, x, y) := \{\lambda \in \Lambda \mid A_\lambda(a) B_\lambda(b) = A_\lambda(x) B_\lambda(y) = -1\}$ and $\Omega_0(a, b, x, y) := \{\lambda \in \Lambda \mid A_\lambda(a) B_\lambda(b) = -A_\lambda(x) B_\lambda(y) = \pm 1\}$ with, $\Lambda = \Omega_0 \cup \Omega_+ \cup \Omega_-$ and the Ω sets disjoint. Hence, $E(a, b) - E(x, y) = \int_{\lambda \in \Lambda} \{A_\lambda(a) B_\lambda(b) - A_\lambda(x) B_\lambda(y)\} \rho_\lambda d\lambda$ only $\lambda \in \Omega_0$ do not cancel. $\Rightarrow E(a, b) - E(x, y) = -2 \int_{\lambda \in \Omega_0(a, b, x, y)} A_\lambda(x) B_\lambda(y) \rho_\lambda d\lambda$. Suppose, $a, b \notin \mathcal{A} \cup \mathcal{B}$ and (a, b) such that $E(a, b) = 0$ then

$$\frac{E(x, y)}{2} = \int_{\lambda \in \Omega_0(a, b, x, y)} A_\lambda(x) B_\lambda(y) \rho_\lambda d\lambda. \quad (2)$$

This is $E_T(x, y)$. Because $E(a, b) = 0$, consistency requires

$$\frac{E(x, y)}{2} = \int_{\lambda \in \Omega_+(a, b, x, y)} \rho_\lambda d\lambda - \int_{\lambda \in \Omega_-(a, b, x, y)} \rho_\lambda d\lambda. \quad (3)$$

This is $E_C(x, y)$. Numerically: $E_T(x, y) \approx E_C(x, y)$.

Suppose, $\rho_\lambda = \rho_{\lambda_1} \rho_{\lambda_2}$ and λ_1 , is assigned to Alice's measuring instrument, λ_2 to Bob's. $\rho_{\lambda_j} = \frac{1}{\sqrt{2}}$ for $\lambda_j \in \Lambda_j$, $\Lambda_j = \{\lambda_j \mid \frac{-1}{\sqrt{2}} \leq \lambda_j \leq \frac{1}{\sqrt{2}}\}$ and zero "elsewhere" ($j = 1, 2$); $\Lambda = \Lambda_1 \times \Lambda_2$. The E_T and E_C transform in

$$E_T(x, y) = \int_{(\lambda_1, \lambda_2) \in \Omega_0(a, b, x, y)} A_{\lambda_1}(x) B_{\lambda_2}(y) \quad (4)$$

and

$$E_C(x, y) = \int_{(\lambda_1, \lambda_2) \in \Omega_+(a, b, x, y)} - \int_{(\lambda_1, \lambda_2) \in \Omega_-(a, b, x, y)} \quad (5)$$

\int represents double integration where necessary. Further, where possible, $d\lambda_1 d\lambda_2$ is suppressed. The A function, for $x \in \mathcal{A}$ is given by

$$A_{\lambda_1}(x) = \begin{cases} \alpha_{\lambda_1}(x), & \lambda_1 \in I(x) \\ \text{sgn}[\zeta(x) - \lambda_1], & \lambda_1 \in (\Lambda_1 \setminus I(x)) \end{cases} \quad (6)$$

Here, $I(1_A) = \{\lambda_1 \mid -\frac{1}{\sqrt{2}} \leq \lambda_1 \leq 1 - \frac{1}{\sqrt{2}}\}$, $I(2_A) = \{\lambda_1 \mid -1 + \frac{1}{\sqrt{2}} \leq \lambda_1 \leq \frac{1}{\sqrt{2}}\}$. For $y \in \mathcal{B}$,

$$B_{\lambda_2}(y) = \begin{cases} \beta_{\lambda_2}(y), & \lambda_2 \in J(y) \\ \text{sgn}[\eta(y) - \lambda_2], & \lambda_2 \in (\Lambda_2 \setminus J(y)) \end{cases} \quad (7)$$

$J(1_B) = \{\lambda_2 \mid -\frac{1}{\sqrt{2}} \leq \lambda_2 \leq 0\}$, $J(2_B) = \Lambda_2 \setminus J(1_B)$, $\text{sgn}(0) = 1$ and $\text{sgn}(x) = \frac{x}{|x|}$, ($x \neq 0$).

Suppose Alice tossed 1_A and Bob 1_B . The α and β are determined by tossing a fair coin. Heads is +1 and tails is -1. Hence, $\Pr_{\text{coins}}\{\alpha_{\lambda_1}(1_A) \beta_{\lambda_2}(1_B) = -1\} > 0$. Moreover, $\forall (x, y) \in \mathcal{Q} \setminus \{(1_A, 1_B)\}$ and $\Pr_{\text{coins}}\{\alpha_{\lambda_1}(x) \beta_{\lambda_2}(y) = 1\} > 0$. In addition to the coins Alice and Bob each hold a 4-sided dice to determine the V and U functions viz. ζ and η in (6) and (7). Suppose that Carrol, by a draw from the model-pool, determines the employed model. She sees $\Pr_{\text{pool}}\{\Omega_+(a, b, 1_A, 1_B) = \emptyset \ \& \ \Omega_-(a, b, 1_A, 1_B) = I(1_A) \times J(1_B)\} > 0$. Hence, $\Pr_{E\text{-space}}\{E_C(1_A, 1_B) = \frac{1}{\sqrt{2}}\} > 0$. E-space is the combination of pool, coins and dices probability spaces. For E_T let us look at $\Pr_{\text{pool}}\{\Omega_0(a, b, 1_A, 1_B) = ((\Lambda_1 \setminus I(1_A)) \times J(1_B)) \cup ((\Lambda_1 \setminus I(1_A)) \times J(2_B)) \cup (I(1_A) \times J(2_B))\} > 0$. Note, $\Omega_0 \cup \Omega_+ \cup \Omega_- = \Lambda$. It follows that

$$E_T(1_A, 1_B) = \int_{(\lambda_1, \lambda_2) \in I(1_A) \times J(2_B)} \alpha u(\lambda_2) + \int_{(\lambda_1, \lambda_2) \in (\Lambda_1 \setminus I(1_A)) \times J(1_B)} \beta v(\lambda_1) + \int_{(\lambda_1, \lambda_2) \in (\Lambda_1 \setminus I(1_A)) \times J(2_B)} v(\lambda_1) u(\lambda_2). \quad (8)$$

Here, $u(\lambda_2) = \text{sgn}[\eta(1_B) - \lambda_2]$ and $v(\lambda_1) = \text{sgn}[\zeta(1_A) - \lambda_1]$. Note, $\int_{I(1_A)} d\lambda_1 = \int_{I(2_A)} d\lambda_1 = 1$ and $\int_{J(1_B)} d\lambda_2 = \int_{J(2_B)} d\lambda_2 = \frac{1}{2}\sqrt{2}$. The more general expression $E_T(x, y) = \alpha U(y) + \frac{\beta}{\sqrt{2}} V(x) + U(y) V(x)$ can subsequently be derived from the previous uv equation. Note, $U(y) = \int_{(\Lambda_2 \setminus J(y))} \text{sgn}[\eta(y) - \lambda_2] d\lambda_2$ and $V(x) = \int_{(\Lambda_1 \setminus I(x))} \text{sgn}[\zeta(x) - \lambda_1] d\lambda_1$. For $V(x)$, $V(1_A) = 2\zeta(1_A) - 1$ and $V(2_A) = 2\zeta(2_A) + 1$. Because of its use as *random* function, $V(x) \in [1 - \sqrt{2}, \sqrt{2} - 1] \approx (-0.4142, 0.4142)$. And, $U(1_B) = 2\eta(1_B) - \frac{1}{\sqrt{2}}$ and $U(2_B) = 2\eta(2_B) + \frac{1}{\sqrt{2}}$ with $U(y) \in \Lambda_2 \approx (-0.7071, 0.7071)$.

The random functions ζ and η translate to 4-sided dices for the sign integrals V and U . We determine the numerical values for U and V below. Suppose $(1_A, 1_B)$ then the E thereof can be rewritten as ($\alpha\beta = -1$) $U(1_B) - \frac{1}{\sqrt{2}} V(1_A) + \alpha U(1_B) V(1_A) = -\frac{\alpha}{\sqrt{2}}$. For $\alpha = 1, \beta = -1$ computation gave $U(1_B) \approx -0.45371$ and $V(1_A) \approx 0.218186$ with error $\delta(U, V) = |U - (V/\sqrt{2}) + \alpha UV - \alpha E_{QM}(a, b)| \approx 9.9 \times 10^{-7}$. Here $E_{QM} = \frac{1}{\sqrt{2}}$. For $\alpha = -1, \beta = 1$, $U(1_B) \approx 0.32760$ and $V(1_A) \approx -0.36691$ with error $\delta(U, V) = 8.0 \times 10^{-7}$. Hence, an approximate consistency in probability: $\Pr_{E\text{-space}}\{E_T(1_A, 1_B) \approx \frac{-1}{\sqrt{2}}\} > 0$. This leads to, $\Pr_{E\text{-space}}\{E(1_A, 1_B) = \frac{-1}{\sqrt{2}}\} > 0$. If Alice tosses 1_A and Bob 2_B then when Alice tosses her α and Bob his β ; $\Pr_{coins}\{\alpha_{\lambda_1}(1_A)\beta_{\lambda_2}(2_B) = 1\} > 0$. Carrol draws from the LHV model pool and has $\Pr_{pool}\{\Omega_+(a, b, 1_A, 2_B) = I(1_A) \times J(2_B) \& \Omega_-(a, b, 1_A, 2_B) = \emptyset\} > 0$. We arrive at $\Pr_{E\text{-space}}\{E_C(1_A, 2_B) = \frac{1}{\sqrt{2}}\} > 0$. In order to determine $E_T(1_A, 2_B)$ the Ω_0 shows $\Pr_{pool}\{\Omega_0(a, b, 1_A, 2_B) = ((\Lambda_1 \setminus I(1_A)) \times J(1_B)) \cup ((\Lambda_1 \setminus I(1_A)) \times J(2_B)) \cup (I(1_A) \times J(1_B))\} > 0$. For $(1_A, 2_B)$, E is ($\alpha\beta = 1$): $U(2_B) + \frac{1}{\sqrt{2}} V(1_A) + \alpha U(2_B) V(1_A) = \frac{\alpha}{\sqrt{2}}$. For $\alpha = 1, \beta = 1$ we have $U(2_B) \approx 0.3001$ and $V(1_A) \approx 0.4042$ with $\delta(U, V) \approx 3.4 \times 10^{-5}$. Here, $E_{QM} = \frac{1}{\sqrt{2}}$. For $\alpha = -1, \beta = -1$ we found $U(2_B) \approx -0.67710$ and $V(1_A) \approx -0.0216$ and $\delta(U, V) \approx 8.0 \times 10^{-7}$. Hence, $\Pr_{E\text{-space}}\{E_T(1_A, 2_B) \approx \frac{1}{\sqrt{2}}\} > 0$, or, $\Pr_{E\text{-space}}\{E(1_A, 2_B) = \frac{1}{\sqrt{2}}\} > 0$. Note that for $(2_A, 1_B)$ and $(2_A, 2_B)$ a similar form for $E(x, y)$ ob-

tains as for $(1_A, 2_B)$. The stress-test amounts to: Alice and Bob determine the setting $(x, y) \in \mathcal{Q}$ and record their spin. At any moment, Alice and Bob may toss α and β coins and throw the dices; $V = (0.218186, -0.36691, 0.4042, -0.0216)$ for Alice and $U = (-0.45371, 0.32760, 0.3001, -0.6771)$ for Bob and make a record using a trial number. Similarly for Carrol's draws from the model-pool.

II. CONCLUSION

For $k = 1, 2, 3, 4, \exists_{n_k \in \{1, \dots, N\}} \exists_{(x, y)_{n_k} \in \mathcal{Q}} \Pr\{E_{T(C)}(x, y)_{n_k} = E_{QM}(x, y)_{n_k} | LHV\} > 0$, and, $(x, y)_n$ the n -th pair of settings x and y . Hence,

$$\Pr\{|S| > 2 | LHV\} = \prod_{k=1}^4 \prod_{(x, y)_{n_k} \in \mathcal{Q}; n_k \in \{1, \dots, N\}} \Pr\{E_{T(C)}(x, y)_{n_k} = E_{QM}(x, y)_{n_k} | LHV\} > 0 \quad (9)$$

and $n_1 \neq n_2, n_2 \neq n_3, n_3 \neq n_4, n_4 \neq n_1$. The urn, four sided dices and the α & β coins connect with non-zero probability the LHV elements of the model. The probability in (9) is based on *per trial* probabilities. In addition, the complete model, i.e. stress test plus LHV part, is Kolmogorovian. Hence, a probability loophole in the CHSH is found. The methodology presented is valid for all settings. The stress-test can be performed at any time. In e.g. experiment [4], a nonzero probability exists that the violation *per quartet* of settings is obtained with LHV. The aim of the experiment is the explanation of the (per quartet) entanglement. Hence, because of (9) the CHSH no-go for LHV is flawed. One can always point at trial numbers where nature *could* have used LHV for $|S| > 2$, because of the result in equation (9). The breach found in CHSH cannot be plugged using LHV impossibilities construed with CHSH principles. Moreover, LHV impossibilities using other means do not take away that LHV may occur in nature in the CHSH type experiments. If (9) does not include LHV, then CHSH does not exclude LHV.

References

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